

Symmetry and degeneracy of the curved Coulomb potential on the S^3 ball

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Abstract: The “curved” Coulomb potential on the S^3 ball, whose isometry group is $SO(4)$, takes the form of a cotangent function, and when added to the four-dimensional squared angular momentum operator, one of the $so(4)$ Casimir invariants, a Hamiltonian is obtained which describes a perturbation of the free geodesic motion that results peculiar in several aspects. The spectrum of such a motion has been studied on various occasions and is known to carry unexpectedly $so(4)$ degeneracy patterns despite the non-commutativity of the perturbation with the Casimir operator. We here suggest an explanation for this behavior in designing a set of operators which close the $so(4)$ algebra and whose Casimir invariant coincides with the Hamiltonian of the perturbed motion at the level of the eigenvalue problem. The above operators are related to the canonical geometric $SO(4)$ generators on S^3 by a non-unitary similarity transformation of the scaling type. In this fashion, we identify a complementary option to the deformed dynamical $so(4)$ Higgs algebra constructed in terms of the components of the ordinary angular momentum and a related Runge-Lenz vector.

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1 Introduction

The three-dimensional hyper-spherical surface, S^3 , embedded in a four-dimensional Euclidean space, E_4 ,

$$S^3 : x_4^2 + x_1^2 + x_2^2 + x_3^2 = R^2, \quad (1)$$

is among the most important geometries in a variety of theoretical physics problems. In first place this curved surface, whose isometry group is $SO(4)$, represents the position space in the celebrated Einstein’s closed Universe and provides one of the major templates in gravitational studies. At the same time, it forms part of the compactified Minkowski spacetime as it emerges upon conformal compactifications in supersymmetric field theories. Further important aspects of the S^3 ball concern its relevance in the effective description of many-body phenomena such as coherent states, fluid-dynamics, [1]-[3], polymer chains [4], Brownian motion [5], and quantum dots [6], all reasons which make studies of physics on S^3 relevant.

The free geodesic motion of a scalar particle on S^3 is described by means of the eigenvalue problem of the squared four-dimensional angular momentum, \mathcal{K} , one of the Casimir invariants of the $so(4)$ isometry algebra. We here focus on a perturbation of this motion by a $\cot \chi$ function of the second polar angle, χ , parametrizing S^3 , which is interesting in so far as it conserves the $SO(4)$ degeneracy patterns in the spectrum of the free geodesic motion, despite its non-commutativity with \mathcal{K} . This property of the cotangent function is quite remarkable indeed and valid not only on S^3 but also on S^2 , and in any higher dimensions. To be specific, also on S^2 the cotangent potential does not remove the $(2\ell + 1)$ -fold degeneracy in the spectrum of the free spherical rigid rotator, despite its non-commutativity with \mathbf{L}^2 [7], [8]. For the explanation of the above degeneracy phenomena, Higgs and Leemon designed in [7], [9] algebras composed by the components of the angular momentum operators, L_i , in the respective external spaces under consideration, and a related Runge-Lenz vector, R_j , designed in analogy to the flat-space one, as known from the H atom problem. This strategy has been successful in explaining the degeneracy phenomena under discussion, despite that L_j and R_i cease to close the respective $so(3)/so(4)$ algebras. Instead, they close algebras which appear as deformations of the respective isometry algebras by terms cubic in the momenta (see also [10] for further details).

We here instead construct a closed $\mathfrak{so}(4)$ algebra for the cotangent perturbed motion on S^3 , which we obtain as a non-unitary similarity transformation of the geometric $\mathfrak{so}(4)$ algebra of the free geodesic motion. Such becomes possible at the level of the $(\mathcal{K} - 2b \cot \chi)$ and \mathcal{K} eigenvalue problems. We build up a matrix similarity transformation which connects the \mathcal{K} and $(\mathcal{K} - 2b \cot \chi)$ carrier spaces, and which happens to be of the dilation (scaling) type. The possibility for constructing such a transformation on S^2 has been indicated in work prior to this [8] and will not be considered here.

We entirely focus on the $\mathfrak{so}(4)$ case which is special by the fact that on S^3 the cotangent function solves the homogeneous Laplace-Beltrami equation, and is a so called harmonic function there. As such, it can be treated along the line of potential theory and the resulting electrodynamics would be Maxwellian, something which does not occur in any other dimension.

The paper is structured as follows. In the next section we present the solutions of the cotangent-perturbed geodesic motion on S^3 in terms of real Romanovski polynomials (reviewed in [11]) and decompose these solutions in the basis of exponentially damped hyper-spherical harmonics. From there we deduce the non-unitary transformation which takes the $\mathfrak{so}(4)$ isometry algebra to a realization that now describes the cotangent-hindered motion. Working at the level of the eigenvalue problems brings as an advantage simplifications by virtue of certain type of recurrence relations among Gegenbauer polynomials. The paper closes with concise conclusions.

2 Particle motion on the S^3 ball

2.1 The free geodesic motion

In polar coordinates, the three-sphere S^3 is parametrized by the azimuthal angle φ , and the two polar angles θ , and χ , according to,

$$S^3 : \quad \begin{aligned} x_4^2 + x_1^2 + x_2^2 + x_3^2 &= R^2, & x_4 &= R \cos \chi, & r &\equiv \sqrt{x_1^2 + x_2^2 + x_3^2} = R \sin \chi, \\ x_3 &= r \cos \theta, & x_1 &= r \sin \theta \cos \varphi, & x_2 &= r \sin \theta \sin \varphi, \end{aligned} \quad (2)$$

where R will be treated as a constant and will be set equal to one for simplicity. The Casimir operator, \mathcal{K} , of the $\mathfrak{so}(4)$ algebra is associated with the squared four-dimensional angular momentum, and is given by [12],

$$\begin{aligned} \mathcal{K} &= \mathbf{L}^2 + \mathbf{N}^2, & L_i &= i\epsilon_{ijk}x_j \frac{\partial}{\partial x_k}, & N_j &= -ix_j \frac{\partial}{\partial x_4} + ix_4 \frac{\partial}{\partial x_j}, \\ \mathbf{N}^2 &= -\frac{1}{\sin \chi} \frac{\partial}{\partial \chi} \sin \chi \frac{\partial}{\partial \chi} + \cot^2 \chi \mathbf{L}^2, \end{aligned} \quad (3)$$

where the components L_i , and N_j of the respective angular momentum, \mathbf{L} , and the Euclidean boost operator, \mathbf{N} , have the property to close the $\mathfrak{so}(4)$ algebra. The quantum-mechanical \mathcal{K} eigenvalue problem is well known and given by,

$$\begin{aligned} \frac{\hbar^2}{2M} \mathcal{K} Y_{K\ell m}(\chi, \theta, \varphi) &= \frac{\hbar^2}{2M} \left[(K+1)^2 - 1 \right] Y_{K\ell m}(\chi, \theta, \varphi), \\ Y_{K\ell m}(\chi, \theta, \varphi) &= \sin^\ell \chi \mathcal{G}_{K-\ell}^{\ell+1}(\cos \chi) Y_\ell^m(\theta, \varphi), \end{aligned} \quad (4)$$

where the constant K determines the value of the four-dimensional angular momentum in the $(K+1)^2$ -dimensional representation space of the hyper-spherical harmonics, $Y_{K\ell m}(\chi, \theta, \varphi)$, $\mathcal{G}_{n=K-\ell}^{\ell+1}(\cos \chi)$ denote the Gegenbauer polynomials, and $Y_\ell^m(\theta, \varphi)$ are the standard three-dimensional spherical harmonics. We have chosen to work in dimensionless units, setting $\hbar = 1$, and $2M = 1$, with M standing for the mass of the particle under consideration. It is obvious that the spectrum of the free geodesic motion on S^3 in (4) is characterized by a $(K+1)^2$ -fold degeneracy of the states, as it should be given the fact that $SO(4)$ is the isometry group of the three-ball. The

immediate conspicuous analogy that comes to ones mind is that modulo the level spacings, the degeneracy patterns of the free geodesic motion on S^3 are same as those appearing in the inverse distance potential problem which shapes the H atom spectrum. Yet, the reasons for the two phenomena are to some extent different, indeed. The common denominator of both degeneracy patterns is an underlying $\text{so}(4)$ symmetry algebra of the associated Hamiltonians. The difference lies in the qualitatively distinct realizations of this very algebra in the respective cases. While the $\text{so}(4)$ algebra of the free geodesic motion on S^3 is purely geometric in the sense that its elements, L_i , and N_j in (3) exclusively depend on the position on the curved surface under consideration, the $\text{so}(4)$ algebra in the inverse distance potential problem is dynamical as its elements need to be defined over the full phase space. Indeed, the $\text{so}(4)$ algebra of the Coulomb potential contains besides the components L_1 , L_2 , and L_3 , of ordinary angular momentum, the three components of the Runge-Lenz vector, R_1 , R_2 , and R_3 [12]. Below, this concept will acquire special importance in the investigation of the perturbed problem.

Combining eqs. (3) and (4), the eigenvalue problem of the squared four-dimensional angular momentum operator becomes,

$$\left[-\frac{1}{\sin^2 \chi} \frac{\partial}{\partial \chi} \sin^2 \chi \frac{\partial}{\partial \chi} + \frac{\mathbf{L}^2}{\sin^2 \chi} \right] Y_{K\ell m}(\chi, \theta, \varphi) = K(K+2) Y_{K\ell m}(\chi, \theta, \varphi), \quad (5)$$

with the hyper-spherical harmonics, $Y_{K\ell m}(\chi, \theta, \varphi)$, being defined in (4) above. In what follows we introduce the short-hand,

$$\mathcal{S}_K^\ell(\chi) = \sin^\ell \chi \mathcal{G}_{K-\ell}^{\ell+1}(\cos \chi), \quad (6)$$

in terms of which the hyper-spherical harmonics equivalently rewrite to,

$$Y_{K\ell m}(\chi, \theta, \varphi) = \mathcal{S}_K^\ell(\chi) Y_\ell^m(\theta, \varphi), \quad K \in [0, \infty), \quad \ell \in [0, K], \quad m \in [-\ell, \ell]. \quad (7)$$

2.2 The “curved” Coulomb potential problem on S^3

For the sake of self-sufficiency of the presentation and fixing notations we here briefly review the exact solutions of the “curved” Coulomb (-like) potential on S^3 , given by,

$$(\mathcal{K} - 2b \cot \chi) \Psi_{K\tilde{\ell}\tilde{m}}(\chi, \theta, \varphi) = \epsilon \Psi_{K\tilde{\ell}\tilde{m}}(\chi, \theta, \varphi). \quad (8)$$

before going to the heart of our study in the next subsection, namely, to their expansions in the basis of the hyper-spherical harmonics. In (8), ϵ is the energy, E , here in dimensionless units, $\epsilon = 2ME/\hbar^2$, and b defines the (equally dimensionless) strength of the cotangent potential. We furthermore admitted in (8) for the possibility that \tilde{m} and $\tilde{\ell}$ may not necessarily be same as m and ℓ in (4), (5).

Upon changing variable in (8) to,

$$\Psi_{K\tilde{\ell}\tilde{m}}(\chi, \theta, \varphi) = \frac{U_K^{\tilde{\ell}}(\chi)}{\sin \chi} Y_{\tilde{\ell}}^{\tilde{m}}(\theta, \varphi), \quad (9)$$

equation (8) transforms into the well known one-dimensional Schrödinger equation with the so called trigonometric Rosen-Morse potential [13], $\mathcal{V}_{\text{RM}}(\chi)$, in reality introduced by Schrödinger [14], [15] as a “curved” Coulomb-, better, Coulomb-like potential [16],

$$\begin{aligned} -\frac{d^2 U_K^{\tilde{\ell}}(\chi)}{d\chi^2} + \mathcal{V}_{\text{RM}}(\chi) U_K^{\tilde{\ell}}(\chi) - (\epsilon + 1) U_K^{\tilde{\ell}}(\chi) &= 0, \\ \mathcal{V}_{\text{RM}}(\chi) &= -2b \cot \chi + \frac{\ell(\ell+1)}{\sin^2 \chi}, \end{aligned} \quad (10)$$

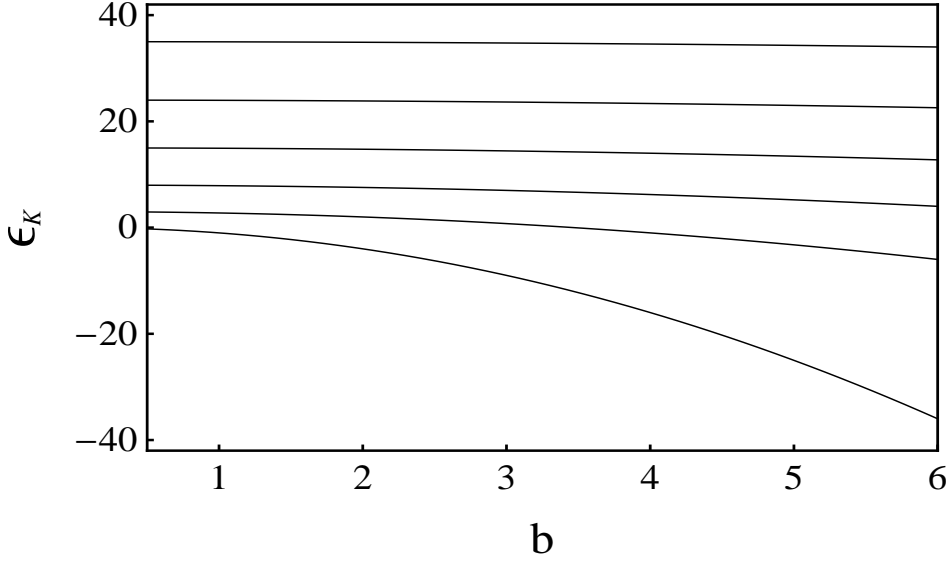


Figure 1: The dependence of the energies, ϵ_K , in (11), of the levels within the “curved” Coulomb potential on the strength b of the $\cot \chi$ perturbation. The figure shows that the cotangent interaction mainly affects the gap between the ground state and its first excitations by increasing it. For moderate values of the potential strength around $b \sim 1$, the higher lying states practically remain unaltered by the perturbation due to the rapid flattening of the exponential factor in (12) with the increase of K . The exponential damping factor furthermore reduces the de-excitation probabilities to the ground state, the most affected being the first excitation.

in depending on the choice for the b value. The latter equation is exactly solvable because it can be reduced to the hyper-geometric differential equation [13], and the spectrum is such that the cotangent perturbation does not remove the $(K + 1)^2$ -fold degeneracy of the free geodesic motion, as visible from the expression for the energy,

$$\epsilon_K + 1 = (K + 1)^2 - \frac{b^2}{(K + 1)^2}. \quad (11)$$

The dependence of the excitation energies in eq. (11) on the strength b of the cotangent perturbation is displayed in Fig. 1.

Among others, the solutions in (9) have been independently reproduced also in [17] in terms of non-classical Romanovski polynomials, $R_n^{\alpha, \beta}(\cot \chi)$, as

$$\begin{aligned} \Psi_{K \tilde{\ell} \tilde{m}}(\chi, \theta, \varphi) &= e^{-\frac{\alpha_K \chi}{2}} \sin^K \chi R_{K-\tilde{\ell}}^{\alpha_K, \beta_K}(\cot \chi) Y_{\tilde{\ell}}^{\tilde{m}}(\theta, \varphi), \\ \beta_K &= -(n + \tilde{\ell}) = -K, \quad \alpha_K = \frac{2b}{K + 1}. \end{aligned} \quad (12)$$

The Romanovski polynomials (reviewed in ref. [11]) satisfy the following differential hyper-geometric equation:

$$(1 + x^2) \frac{d^2 R_n^{\alpha, \beta}}{dx^2} + 2 \left(\frac{\alpha}{2} + \beta x \right) \frac{d R_n^{\alpha, \beta}}{dx} - n(2\beta + n - 1) R_n^{\alpha, \beta} = 0. \quad (13)$$

They are obtained from the following weight function,

$$\omega^{\alpha, \beta}(x) = (1 + x^2)^{\beta-1} \exp(-\alpha \cot^{-1} x), \quad (14)$$

by means of the Rodrigues formula,

$$R_n^{\alpha,\beta}(x) = \frac{1}{\omega^{\alpha,\beta}(x)} \frac{d^n}{dx^n} [(1+x^2)^n \omega^{\alpha,\beta}(x)]. \quad (15)$$

Upon introducing the short-hand,

$$\psi_K^{\tilde{\ell}}(\chi) = \sin^K \chi R_{K-\tilde{\ell}}^{\alpha_K, \beta_K}(\cot \chi), \quad (16)$$

one arrives at the final form of the solutions to eq. (8),

$$\Psi_{K\tilde{\ell}\tilde{m}}(\chi, \theta, \varphi) = e^{-\frac{\alpha_K \chi}{2}} \psi_K^{\tilde{\ell}}(\chi) Y_{\tilde{\ell}}^{\tilde{m}}(\theta, \varphi). \quad (17)$$

2.3 The perturbed motion in terms of damped hyper-spherical harmonics

The goal of the present section is to find finite decompositions of the exact solutions of the “curved”Coulomb(-like) potential on S^3 in the basis of the canonical hyper-spherical harmonics describing the free geodesic motion. We begin with seeking to relate the χ dependent parts, $\psi_K^{\tilde{\ell}}(\chi)$, and $\mathcal{S}_K^{\ell}(\chi)$, of the wave functions of the respective perturbed, and free motions in (16), and (6), as

$$\psi_K^{\tilde{\ell}}(\chi) = \sum_{\ell=\tilde{\ell}}^K C_{\ell} \mathcal{S}_K^{\ell}(\chi) = \sum_{\ell=\tilde{\ell}}^K C_{\ell} \sin^{\ell} \chi \mathcal{G}_{K-\ell}^{\ell+1}(\cos \chi). \quad (18)$$

The latter equation in fact represents a new ansatz for constructing $\psi_K^{\tilde{\ell}}(\chi)$ in (16), and it is indeed possible to write down series of conditions which fix the constants C_{ℓ} . However, one can equally well take advantage of already knowing $\psi_K^{\tilde{\ell}}(\chi)$, and encounter the expansion coefficients in (18) using the orthogonality properties of the hyper-spherical harmonics. Both ways are eligible. We here opt for the second one, and encounter the expansions given in Table 1 below. Substituting them in eq. (17), and making use of the identity,

$$\mathcal{S}_K^{\ell}(\chi) \equiv \frac{e^{-im\varphi}}{P_{\ell}^m(\cos \theta)} Y_{K\ell m}(\chi, \theta, \varphi), \quad (19)$$

allows to cast the decompositions in the following matrix form,

$$\begin{pmatrix} \Psi_{100}(\chi, \theta, \varphi) \\ \Psi_{11\tilde{m}}(\chi, \theta, \varphi) \end{pmatrix} = e^{-\frac{\alpha_1 \chi}{2}} \mathbf{A}_1(\theta, \varphi) \begin{pmatrix} Y_{100}(\chi, \theta, \varphi) \\ Y_{111}(\chi, \theta, \varphi) \end{pmatrix}, \quad (20)$$

$$\begin{pmatrix} \Psi_{200}(\chi, \theta, \varphi) \\ \Psi_{21\tilde{m}_1}(\chi, \theta, \varphi) \\ \Psi_{22\tilde{m}_2}(\chi, \theta, \varphi) \end{pmatrix} = e^{-\frac{\alpha_2 \chi}{2}} \mathbf{A}_2(\theta, \varphi) \begin{pmatrix} Y_{200}(\chi, \theta, \varphi) \\ Y_{211}(\chi, \theta, \varphi) \\ Y_{222}(\chi, \theta, \varphi) \end{pmatrix}, \quad (21)$$

$$\begin{pmatrix} \Psi_{300}(\chi, \theta, \varphi) \\ \Psi_{31\tilde{m}_1}(\chi, \theta, \varphi) \\ \Psi_{32\tilde{m}_2}(\chi, \theta, \varphi) \\ \Psi_{33\tilde{m}_3}(\chi, \theta, \varphi) \end{pmatrix} = e^{-\frac{\alpha_3 \chi}{2}} \mathbf{A}_3(\theta, \varphi) \begin{pmatrix} Y_{300}(\chi, \theta, \varphi) \\ Y_{311}(\chi, \theta, \varphi) \\ Y_{322}(\chi, \theta, \varphi) \\ Y_{333}(\chi, \theta, \varphi) \end{pmatrix}, \quad (22)$$

where we chose $m = \ell$ in (19). The matrices $\mathbf{A}_K(\theta, \varphi)$ operate on the space of exponentially scaled (damped) pseudo-spherical harmonics,

$$\tilde{Y}_{K\ell m}(\chi, \theta, \varphi) = e^{-\frac{\alpha_K \chi}{2}} Y_{K\ell m}(\chi, \theta, \varphi). \quad (23)$$

The explicit expressions for the $\mathbf{A}_K(\theta, \varphi)$ matrices for the lowest K values are given by,

$$\mathbf{A}_1(\theta, \varphi) = \begin{pmatrix} 1 & \frac{be^{-i\varphi}}{P_1^1} \\ 0 & \frac{e^{i(\bar{m}-1)\varphi} P_1^{\bar{m}}}{P_1^1} \end{pmatrix}, \quad (24)$$

$$\mathbf{A}_2(\theta, \varphi) = \begin{pmatrix} 1 & \frac{be^{-i\varphi}}{P_1^1} & \frac{(2b)^2}{3^2} \frac{e^{-2i\varphi}}{P_2^2} \\ 0 & \frac{e^{i(\bar{m}_1-1)\varphi} P_1^{\bar{m}_1}}{P_1^1} & \frac{2b}{3} \frac{e^{i(\bar{m}_1-2)\varphi} P_1^{\bar{m}_1}}{P_2^2} \\ 0 & 0 & \frac{e^{i(\bar{m}_2-2)\varphi} P_2^{\bar{m}_2}}{P_2^2} \end{pmatrix}, \quad (25)$$

$$\mathbf{A}_3(\theta, \varphi) = \begin{pmatrix} 1 & \frac{9be^{-i\varphi}}{10P_1^1} & \frac{b^2 e^{-i2\varphi}}{2P_2^2} & \left(\frac{b^3}{8} - \frac{2b}{5}\right) \frac{e^{-i3\varphi}}{P_3^3} \\ 0 & \frac{e^{i(\bar{m}_1-1)\varphi} P_1^{\bar{m}_1}}{P_1^1} & \frac{5be^{i(\bar{m}_1-2)\varphi} P_1^{\bar{m}_1}}{6P_2^2} & \frac{b^2 e^{i(\bar{m}_1-3)\varphi} P_1^{\bar{m}_1}}{2^2 P_3^3} \\ 0 & 0 & \frac{e^{i(\bar{m}_2-2)\varphi} P_2^{\bar{m}_2}}{P_2^2} & \frac{be^{i(\bar{m}_2-3)\varphi} P_2^{\bar{m}_2}}{2P_3^3} \\ 0 & 0 & 0 & \frac{e^{i(\bar{m}_3-3)\varphi} P_3^{\bar{m}_3}}{P_3^3} \end{pmatrix}, \quad (26)$$

where we have dropped the $\cos \theta$ argument of the associated Legendre functions, $P_\ell^\ell(\cos \theta)$, for the sake of simplifying the notations. In noticing that $P_\ell^\ell(\cos \theta) \sim \sin^\ell \theta$, we observe that the matrices connecting the representation functions of the perturbed and free motions on S^3 are invertible in the entire open interval, $\theta \in (0, \pi)$, being singular only at the poles. Invertible matrices $\mathbf{A}_K(\theta, \varphi)$ at the poles can be obtained by replacing $P_l^l(\cos \theta)$ in eqs. (24)–(26), through either $P_l^0(\cos 0)$ (“North” pole), or $P_l^0(\cos \pi)$ (“South” pole), this because the special values of the Legendre polynomials are finite at the ends of the interval under consideration. On S^2 the related matrices depend on the azimuthal angle φ alone and are completely free from singularities.

2.4 Scaling similarity transformation of the geometric $\mathfrak{so}(4)$ algebra on S^3 to the algebra of the perturbed motion

Substituting (18) in (8) and dragging the exponential factor from the very right to the very left results in

$$\begin{aligned} e^{-\frac{\alpha_K \chi}{2}} \left(-\frac{1}{\sin^2 \chi} \frac{\partial}{\partial \chi} \sin^2 \chi \frac{\partial}{\partial \chi} + \frac{\tilde{\ell}(\tilde{\ell}+1)}{\sin^2 \chi} - \frac{\alpha_K^2}{4} + \alpha_K \mathbf{D}_K \right) \sum_{\ell=\tilde{\ell}}^K C_\ell \mathcal{S}_K^\ell(\chi) \\ = \left(-\frac{\alpha_K^2}{4} + K(K+2) \right) \sum_{\ell=\tilde{\ell}}^K C_\ell e^{-\frac{\alpha_K \chi}{2}} \mathcal{S}_K^\ell(\chi). \end{aligned} \quad (27)$$

Here, we introduced the differential operator \mathbf{D}_K as,

$$\mathbf{D}_K = \left(\frac{\partial}{\partial \chi} - K \cot \chi \right), \quad (28)$$

and used $\alpha_K = 2b/(K+1)$. For $\tilde{\ell}$ taking the maximal value, $\tilde{\ell} = K$, the operator \mathbf{D}_K coincides with the raising operator of the $\mathfrak{so}(4)$ algebra and nullifies $\mathcal{S}_K^K(\chi) \sim \sin^K \chi$. In effect, one encounters the identity,

$$\begin{aligned} [\mathcal{K} - 2b \cot \chi] \Psi_{KK\bar{m}}(\chi, \theta, \varphi) &= \left[e^{-\frac{\alpha_K \chi}{2}} \left(\mathcal{K} - \frac{\alpha_K^2}{4} \right) e^{\frac{\alpha_K \chi}{2}} \right] \Psi_{KK\bar{m}}(\chi, \theta, \varphi) \\ &= \left(K(K+2) - \frac{\alpha_K^2}{4} \right) \Psi_{KK\bar{m}}(\chi, \theta, \varphi). \end{aligned} \quad (29)$$

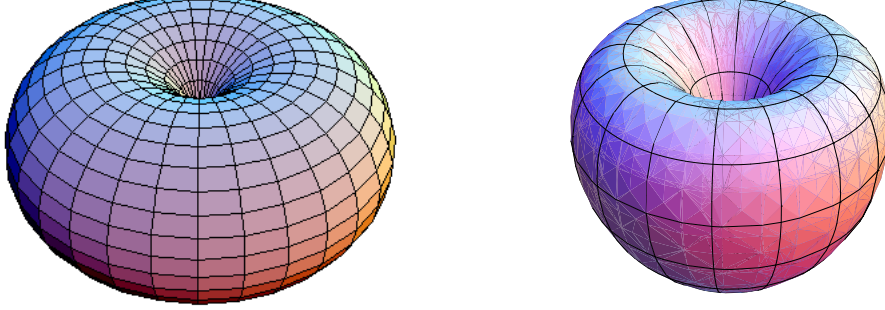


Figure 2: The regular spherical harmonic $|Y_{111}(\chi, 0, \varphi)|$ (left) versus the damped, $|\tilde{Y}_{111}(\chi, 0, \varphi)|$, (right) for $b=2$.

The latter expression shows that up to a non-unitary scaling transformation, and a shift by a constant, the eigenvalue problem of the cotangent perturbed geodesic motion on S^3 for $|KK\tilde{m} >$ results equivalent to the eigenvalue problem of the free geodesic motion. Such in fact is valid for any set of the quantum numbers, $|K\tilde{\ell}\tilde{m} >$, this by virtue of the following recurrence relations among $S_K^l(\chi)$ functions, which translate into recurrence relations among Gegenbauer polynomials,

$$\begin{aligned} D_1 S_1^0(\chi) &= 2 \csc^2 \chi S_1^1(\chi), & D_2 S_2^1(\chi) &= 4 \csc^2 \chi S_2^2(\chi), \\ D_2 S_2^0(\chi) &= 2 \csc^2 \chi S_2^1(\chi), & D_3 S_3^2(\chi) &= 6 \csc^2 \chi S_3^3(\chi), \\ & & D_3 S_3^1(\chi) &= \frac{20}{3} \csc^2 \chi S_3^2(\chi), \\ D_K S_K^K(\chi) &= 0, \quad \forall K, \text{ etc.}, & S_K^\ell(\chi) &= \sin^\ell \chi \mathcal{G}_{K-\ell}^{\ell+1}(\cos \chi). \end{aligned} \quad (30)$$

In order to illustrate the rôle of the recurrence relations we here present the simple example of $\Psi_{100}(\chi, \theta, \varphi)$. In this case, and according to Table 1, equation (18) reduces to

$$\begin{aligned} \psi_1^0(\chi) &= S_1^0(\chi) + b S_1^1(\chi), \\ S_1^0(\chi) &= -2 \sin \chi \cot \chi, & S_1^1(\chi) &= \sin \chi. \end{aligned} \quad (31)$$

Upon substitution of (31) in (27), it is straightforward showing that by virtue of the first relation in (30), the $\alpha_1 D_1 S_1^0(\chi)$ term produces the exact centrifugal term of the second component, $b S_1^1(\chi)$, so that the net action of $(\mathcal{K} - 2b \cot \chi)$ on $e^{-\frac{\alpha_1 \chi}{2}} \psi_1^0(\chi)$ becomes,

$$\begin{aligned} \left(\mathcal{K} - 2b \cot \chi \right) e^{-\frac{\alpha_1 \chi}{2}} \left(S_1^0(\chi) + b S_1^1(\chi) \right) &= \left[e^{-\frac{\alpha_1 \chi}{2}} \left(\mathcal{K} - \frac{\alpha_1^2}{4} \right) e^{\frac{\alpha_1 \chi}{2}} \right] \left[e^{-\frac{\alpha_1 \chi}{2}} \left(S_1^0(\chi) + b S_1^1(\chi) \right) \right] \\ &= \left(3 - \frac{\alpha_1^2}{4} \right) e^{-\frac{\alpha_1 \chi}{2}} \psi_1^0(\chi). \end{aligned} \quad (32)$$

In Figs. 2, and 3, the modification of the shapes of hyper-spherical and damped hyper-spherical harmonics corresponding to the case in eq. (31), are shown for illustrative purposes.

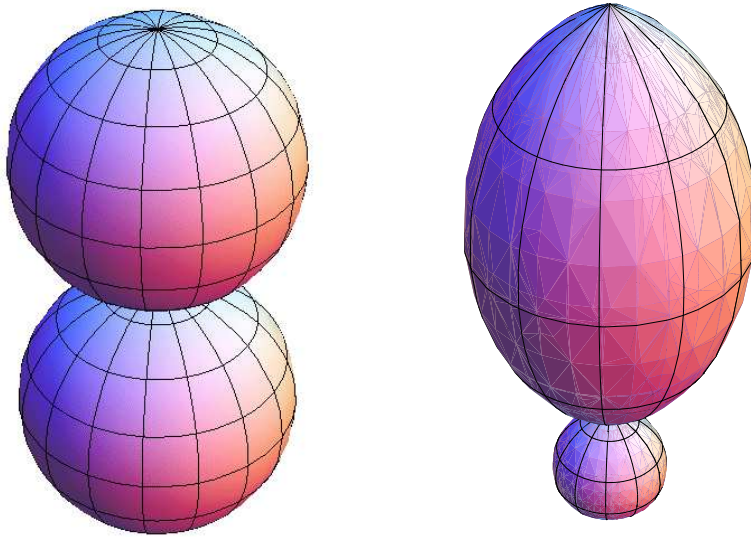


Figure 3: The regular hyper-spherical harmonic $|Y_{100}(\chi, 0, \varphi)|$ (left) versus the damped, $|\tilde{Y}_{100}(\chi, 0, \varphi)|$, (right) for $b=0.45$.

Therefore, at the general level of the total wave functions, one finds,

$$\begin{aligned} [\mathcal{K} - 2b \cot \chi] \Psi_{100}(\chi, \theta, \varphi) &= \left[e^{-\frac{\alpha_K \chi}{2}} \left(\mathcal{K} - \frac{\alpha_K^2}{4} \right) e^{\frac{\alpha_K \chi}{2}} \right] \Psi_{100}(\chi, \theta, \varphi) \\ &= \left(K(K+2) - \frac{\alpha_K^2}{4} \right) \Psi_{100}(\chi, \theta, \varphi). \end{aligned} \quad (33)$$

In fact, the recurrence relations in (30) guarantee validity of,

$$\left(\frac{\tilde{\ell}(\tilde{\ell}+1)}{\sin^2 \chi} + \alpha_K \mathbf{D}_K \right) \sum_{\ell=\tilde{\ell}}^{\ell=K} C_\ell \mathcal{S}_K^\ell = \sum_{\ell=\tilde{\ell}}^{\ell=K} \frac{\ell(\ell+1)}{\sin^2 \chi} C_\ell \mathcal{S}_K^\ell, \quad (34)$$

which, upon substitution in (27), allows to generalize the transformation in (29) and (33) to any $\Psi_{K\tilde{\ell}\tilde{m}}(\chi, \theta, \varphi)$. In consequence, the similarity transformation between the $(\mathcal{K} - 2b \cot \chi)$ and the \mathcal{K} eigenvalue problems can be cast in the following matrix form,

$$\begin{aligned} (\mathcal{K} - 2b \cot \chi) \mathbf{X}_K(\chi, \theta, \varphi) &= \left[e^{-\frac{\alpha_K \chi}{2}} \mathbf{A}_K(\theta, \varphi) \left(\mathcal{K} - \frac{\alpha_K^2}{4} \right) e^{\frac{\alpha_K \chi}{2}} \mathbf{A}_K^{-1}(\theta, \varphi) \right] \mathbf{X}_K(\chi, \theta, \varphi) \\ \mathbf{X}_K(\chi, \theta, \varphi) &= \begin{pmatrix} \Psi_{K00}(\chi, \theta, \varphi) \\ \Psi_{K1\tilde{m}_1}(\chi, \theta, \varphi) \\ \dots \\ \Psi_{KK\tilde{m}_K}(\chi, \theta, \varphi) \end{pmatrix}. \end{aligned} \quad (35)$$

The latter equation shows that the eigenvalue problem of the cotangent perturbed motion on S^3 can be represented as the eigenvalue problem of a similarity-transformed Casimir operator, \mathcal{K} , of the ordinary $\text{so}(4)$ isometry algebra, shifted by a constant (within the representation space of interest). The transformation is non-unitary and of the dilation type. As long as the similarity transformation of the Casimir invariant of the geometric algebra can be viewed as the result of subjecting the $\text{so}(4)$ elements L_i , and N_j in (7) to same transformation, we here have designed a representation of the geometric $\text{so}(4)$ algebra for a “curved” Coulomb potential on S^3 .

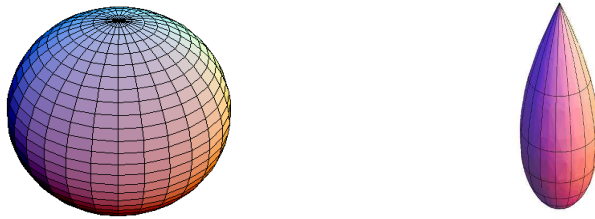


Figure 4: Deformation of the spherical metric, $|Y_{000}(\chi, 0, \varphi)|$, (left) versus the exponentially scaled one, $|\tilde{Y}_{000}(\chi, 0, \varphi)|$ (right) for $b=1$.

3 Conclusions

The clue of the present study is that modulo a shift by a constant, a non-unitary scaling similarity transformation converts the Casimir invariant of the $\text{so}(4)$ isometry algebra of the three-ball S^3 into the Coulomb potential problem

there. The transformation, presented in (35), was concluded from decomposing the eigenfunctions of the “curved” Coulomb potential problem in the bases of exponentially scaled (damped) hyper-spherical harmonics in (20)–(23). It was furthermore shown to emerge by virtue of specific recurrence relations (30) among Gegenbauer polynomials.

The particle motion on S^3 considered here had the peculiarity that the perturbation left the degeneracy patterns characterizing the free motion intact, though the isometry group symmetry has been broken. The breaking of the initial $SO(4)$ rotational invariance by the “curved” Coulomb potential happened at the level of the representation functions, thus avoiding the more severe breakdown at the level of the algebra through deformations [7], [9], [10], [18]. This subtle type of symmetry breaking is visualized in Fig. 4 by the deformation of the metric of the S^3 sphere through the damping exponential factor under consideration. It might be interesting to notice, that such a metric deformation emerges also in effect of a conformal symmetry breaking by the dilaton mass [19]. The method proposed can easily be extended towards hyperbolic spaces through complexification of the polar angle χ in (2) as $\chi \rightarrow i\chi$. Simultaneously changing $b \rightarrow -ib$ in (10) takes one to the Coulomb problem on a hyperboloid. Quantum problems on hyperbolic spaces, pioneered by Dirac [20] through his consideration of the electron wave equation in the De-Sitter universe, acquire importance as well within the context of pure relativistic descriptions of bound systems, as within the context of gravitational studies.

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K	$\tilde{\ell}$	$\tilde{\ell} \leq \ell \leq K$	$\psi_K^{\tilde{\ell}}(\chi) = \sum_{\ell=\tilde{\ell}}^K C_\ell \mathcal{S}_K^\ell(\chi)$
0	0	0	$\psi_0^0(\chi) = \mathcal{S}_0^0(\chi)$
1	0	0, 1	$\psi_1^0(\chi) = \mathcal{S}_1^0(\chi) + b \mathcal{S}_1^1(\chi)$
1	1	1	$\psi_1^1(\chi) = \mathcal{S}_1^1(\chi)$
2	0	0, 1, 2	$\psi_2^0(\chi) = \mathcal{S}_2^0(\chi) + b \mathcal{S}_2^1(\chi) + (\frac{2b}{3})^2 \mathcal{S}_2^2(\chi)$
2	1	1, 2	$\psi_2^1(\chi) = \mathcal{S}_2^1(\chi) + \frac{2}{3} b \mathcal{S}_2^2(\chi)$
2	2	2	$\psi_2^2(\chi) = \mathcal{S}_2^2(\chi)$
3	0	0, 1, 2, 3	$\psi_3^0(\chi) = \mathcal{S}_3^0(\chi) + \frac{9}{10} b \mathcal{S}_3^1(\chi) + \frac{b^2}{2} \mathcal{S}_3^2(\chi) + \left(\frac{b^3}{8} - \frac{2b}{5}\right) \mathcal{S}_3^3(\chi)$
3	1	1, 2, 3	$\psi_3^1(\chi) = \mathcal{S}_3^1(\chi) + \frac{5}{6} b \mathcal{S}_3^2(\chi) + (\frac{b}{2})^2 \mathcal{S}_3^3(\chi)$
3	2	2, 3	$\psi_3^2(\chi) = \mathcal{S}_3^2(\chi) + \frac{b}{2} \mathcal{S}_3^3(\chi)$
3	3	3	$\psi_3^3(\chi) = \mathcal{S}_3^3(\chi)$

Table 1: The $\psi_K^\ell(\chi)$ parts of the solutions of the perturbed motion in eq. (16) in the basis of the free motion in eq. (6).